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2002 J. Phys. A: Math. Gen. 35 9911

(http://iopscience.iop.org/0305-4470/35/46/313)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 9911–9922 PII: S0305-4470(02)39273-4

Electromagnetic multipoles in Cartesian coordinates

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Received 10 July 2002
Published 7 November 2002
Online at stacks.iop.org/JPhysA/35/9911

Abstract
A general procedure for the reduction of Cartesian electric and magnetic multipole tensors to symmetric traceless ones is presented. In the static case this procedure applies independently in each order of the multipole expansion, whereas in the dynamic case the process of reduction is recursive. The expressions of the reduced multipole tensors differ in the dynamic case from the static ones by toroidal moments and mean radii of various orders.

PACS numbers: 03.50.De, 41.20.−q

1. Introduction

The multipole expansion of the vector, $\boldsymbol{A}$, and scalar, Φ, potentials is an important tool for the study of the electromagnetic field generated by charge distributions, including many applications. A full and systematic treatment in spherical coordinates is given in most textbooks (see, e.g., [1]). In Cartesian coordinates, the problem is presented fully only in the static case [2–8], while the treatment of the dynamic case is given only for the lowest order multipoles. The main difficulty in the case of Cartesian coordinates is the procedure of reduction of the nth-order multipole tensors to symmetric traceless ones to obtain in this way a description of multipoles in terms of irreducible rotation group tensors. A full treatment of the multipole expansion in the Cartesian case would be warranted at least by concerns about methodological completeness.

In this paper we present the general procedure for the reduction of multipole tensors represented by Cartesian coordinate components to symmetric traceless ones in the static and dynamic cases. In section 2, we present basic formulae for multipole expansions. We give some examples of the use of these formulae for treating fundamental questions of electrodynamics such as multipole expansions of charge and current distributions and radiation fields. In section 3, following an earlier work of the author [6], the reduction procedure in the static case is presented. Some examples of utilization of the reduced multipole tensors to represent the static fields and interaction energies are given. The reduction of the electric and magnetic multipole tensors in the dynamic case is treated in section 4. The transformations implied in the reduction procedure are defined such that the electromagnetic potentials are altered only by gauge transformations. This implies a specific feature of the dynamic case: the redefinitions of the multipole tensors in the lower $n < N$ orders induced by the reduction of tensors in a given order N. Summarizing the results for the dynamic case, in section 5 is given a simple example of reduction beginning from the electric octupole, and it is shown that the redefinitions of the electric dipole and quadrupole moments are obtained by adding the time derivatives of toroid dipole and quadrupole moments, respectively.

2. Basic formulae for the multipole expansions

Let us consider charge $\rho(\boldsymbol{r}, t)$ and current $\boldsymbol{j}(\boldsymbol{r}, t)$ distributions having supports included in a finite domain $\mathcal{D}$. Choosing the origin O of the Cartesian coordinates in $\mathcal{D}$, and using the notation $\boldsymbol{e}_i$ for the orthogonal unit vectors along the axes, the retarded scalar and vector potentials at a point outside $\mathcal{D}$, $\boldsymbol{r} = x_i \boldsymbol{e}_i$, are

$$\Phi(\boldsymbol{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\boldsymbol{\xi}, t - R/c)}{R} \,\mathrm{d}^3\xi \qquad \boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{j}(\boldsymbol{\xi}, t - R/c)}{R} \,\mathrm{d}^3\xi \tag{1}$$

where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{\xi}$. The Taylor series expansion of the function $f(R)$ is

$$f(R) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \xi_{i_1} \cdots \xi_{i_n} \partial_{i_1 \ldots i_n} f(r) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \boldsymbol{\xi}^n \cdot \nabla^n f(r) \tag{2}$$

where

$$\partial_{i_1 \ldots i_n} = \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_n}}$$

and $\boldsymbol{a}^n$ is the n-fold tensorial product $\boldsymbol{a} \otimes \cdots \otimes \boldsymbol{a} : (\boldsymbol{a} \otimes \cdots \otimes \boldsymbol{a})_{i_1 \ldots i_n} = a_{i_1} \cdots a_{i_n}$. Denoting by $\mathsf{T}^{(n)}$ an n-order tensor, $\mathsf{A}^{(n)} \cdot \mathsf{B}^{(m)}$ is an $|n - m|$-order tensor with the components

$$(\mathsf{A}^{(n)} \cdot \mathsf{B}^{(m)})_{i_1 \ldots i_{|n-m|}} = \begin{cases} A_{i_1 \ldots i_{n-m} j_1 \ldots j_m} B_{j_1 \ldots j_m} & n > m \\ A_{j_1 \ldots j_n} B_{j_1 \ldots j_n} & n = m\,. \\ A_{j_1 \ldots j_n} B_{j_1 \ldots j_n i_1 \ldots i_{m-n}} & n < m \end{cases}$$

Substituting equation (2) into (1) we get the well-known multipole expansion of the scalar potential

$$\Phi(\boldsymbol{r}, t) = \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \nabla^n \cdot \left[\frac{1}{r} \mathsf{P}^{(n)}(t_0)\right] \qquad t_0 = t - \frac{r}{c}. \tag{3}$$

Here,

$$\mathsf{P}^{(n)}(t) = \int_{\mathcal{D}} \boldsymbol{\xi}^n \rho(\boldsymbol{\xi}, t) \,\mathrm{d}^3\xi$$

is the nth-order electric multipole tensor.

For the vector potential, we get

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \partial_{i_1 \ldots i_n} \left[\frac{1}{r} \int_{\mathcal{D}} \xi_{i_1} \cdots \xi_{i_n} \boldsymbol{j}(\boldsymbol{\xi}, t - r/c) \,\mathrm{d}^3\xi\right]. \tag{4}$$

Equation (4) is similar to (3) for the scalar potential. The definition of the magnetic multipole tensors implied by equation (4)

$$\mu_{i_1 \ldots i_n} = \int_{\mathcal{D}} \xi_{i_1} \cdots \xi_{i_{n-1}} j_{i_n}(\boldsymbol{\xi}, t) \,\mathrm{d}^3\xi$$

differs from the usual definition of the magnetic multipole moments

$$\mathsf{M}^{(n)} = \frac{n}{n+1}\int_{\mathcal{D}} \boldsymbol{\xi}^n \times \boldsymbol{j}(\boldsymbol{\xi}, t)\,\mathrm{d}^3\xi \tag{5}$$

where we use the notation

$$(\mathsf{T}^{(n)} \times \boldsymbol{a})_{i_1\ldots i_n} = \varepsilon_{i_n i j}\mathsf{T}_{i_1\ldots i_{n-1} i}a_j.$$

More explicitly,

$$\mathsf{M}_{i_1\ldots i_n} = \frac{n}{n+1}\varepsilon_{i_n kl}\int_{\mathcal{D}} \xi_{i_1}\cdots\xi_{i_{n-1}}\xi_k j_l\,\mathrm{d}^3\xi = \frac{n}{n+1}\int_{\mathcal{D}} \xi_{i_1}\cdots\xi_{i_{n-1}}(\boldsymbol{\xi}\times\boldsymbol{j})_{i_n}\,\mathrm{d}^3\xi.$$

We can recover the latter by generalizing a procedure given in [5] for the static case, which was used in [7] for representing the multipole expansion (4)

$$\frac{4\pi}{\mu_0}\boldsymbol{A}(\boldsymbol{r}, t) = \nabla \times \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\nabla^{n-1}\cdot\left[\frac{1}{r}\mathsf{M}^{(n)}(t_0)\right] + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\nabla^{n-1}\cdot\left[\frac{1}{r}\dot{\mathsf{P}}^{(n)}(t_0)\right]. \tag{6}$$

Here we use the superdot notation for the time derivatives.

We will illustrate the usefulness of expansions (3) and (6) by two examples connected to fundamental questions of electrodynamics.

The multipole expansions of the charge and current densities follow from expansions (3) and (6) and

$$\rho(\boldsymbol{r}, t) = -\varepsilon_0\,\square\,\Phi(\boldsymbol{r}, t) \qquad \boldsymbol{j}(\boldsymbol{r}, t) = -\frac{1}{\mu_0}\,\square\,\boldsymbol{A}(\boldsymbol{r}, t)$$

$$\square\,\frac{f(t - r/c)}{r} = -4\pi f(t - r/c)\delta(\boldsymbol{r}) \qquad \left(\square = \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right).$$

Here $\delta(\boldsymbol{r})$ is the Dirac function.

Defining the electric and magnetic polarization intensity vectors,

$$\begin{aligned}\boldsymbol{\mathcal{P}}(\boldsymbol{r}, t) &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\nabla^{n-1}\cdot[\mathsf{P}^{(n)}(t)\delta(\boldsymbol{r})]\\ \boldsymbol{\mathcal{M}}(\boldsymbol{r}, t) &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\nabla^{n-1}\cdot[\mathsf{M}^{(n)}(t)\delta(\boldsymbol{r})]\end{aligned} \tag{7}$$

the following relations can be obtained:

$$\rho = Q\delta(\boldsymbol{r}) - \nabla\cdot\boldsymbol{\mathcal{P}} \qquad \boldsymbol{j} = \nabla\times\boldsymbol{\mathcal{M}} + \frac{\partial\boldsymbol{\mathcal{P}}}{\partial t} \tag{8}$$

where Q is the total electric charge.

Considering in equation (8) $Q = 0$, by a straightforward calculation [7], one may demonstrate the relations

$$\begin{aligned}\Phi(\boldsymbol{r}, t) &= -\frac{1}{4\pi\varepsilon_0}\int_{\mathcal{D}}\frac{1}{R}[\nabla_\xi\cdot\boldsymbol{\mathcal{P}}(\boldsymbol{\xi}, t')]_{t'=t-R/c}\,\mathrm{d}^3\xi\\ &= \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{D}}\left[\frac{\boldsymbol{R}\cdot\boldsymbol{\mathcal{P}}(\boldsymbol{\xi}, t - R/c)}{R^3} + \frac{\boldsymbol{R}\cdot\dot{\boldsymbol{\mathcal{P}}}(\boldsymbol{\xi}, t - R/c)}{cR^2}\right]\mathrm{d}^3\xi\\ \boldsymbol{A}(\boldsymbol{r}, t) &= \frac{\mu_0}{4\pi}\int_{\mathcal{D}}\frac{1}{R}[\nabla_\xi\times\boldsymbol{\mathcal{M}}(\boldsymbol{\xi}, t')]_{t'=t-R/c}\,\mathrm{d}^3\xi + \frac{\mu_0}{4\pi}\int_{\mathcal{D}}\frac{1}{R}\dot{\boldsymbol{\mathcal{P}}}(\boldsymbol{\xi}, t - R/c)\,\mathrm{d}^3\xi\\ &= -\frac{\mu_0}{4\pi}\int_{\mathcal{D}}\left[\frac{\boldsymbol{R}\times\boldsymbol{\mathcal{M}}(\boldsymbol{\xi}, t - R/c)}{R^3} + \frac{\boldsymbol{R}\times\dot{\boldsymbol{\mathcal{M}}}(\boldsymbol{\xi}, t - R/c)}{cR^2}\right]\mathrm{d}^3\xi\\ &\quad + \frac{\mu_0}{4\pi}\int_{\mathcal{D}}\frac{1}{R}\dot{\boldsymbol{\mathcal{P}}}(\boldsymbol{\xi}, t - R/c)\,\mathrm{d}^3\xi.\end{aligned}$$

Taking into account the expressions of the potentials corresponding to point electric and magnetic dipoles, these equations prove the following theorem of equivalence: *if a neutral distribution is represented by relations (8), then this distribution is equivalent to spatial electric and magnetic dipole distributions with the densities $\mathcal{P}$ and $\mathcal{M}$.* As is well known [9], relations (8) are very useful for understanding the microscopic nature of the macroscopic electric and magnetic polarizations of the matter.

The second example is a fundamental formula for the radiation of a localized charge and current distribution which can be obtained from expansions (3) and (6). Since only the terms of order $1/r$, $r \to \infty$ from the expansion of the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ contribute to the radiation of a localized charge and current distribution, we need to consider only [10]

$$\boldsymbol{A}_{\mathrm{rad}}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\frac{1}{r}\int_{\mathcal{D}} \boldsymbol{j}(\boldsymbol{\xi}, t - R/c)\,\mathrm{d}^3\xi.$$

Thus, the parts of $\boldsymbol{E}$ and $\boldsymbol{B}$ which contribute to the radiation are

$$\boldsymbol{B}_{\mathrm{rad}}(\boldsymbol{r},t) = \frac{1}{c}\left[\frac{\partial \boldsymbol{A}_{\mathrm{rad}}}{\partial t}(\boldsymbol{r},t)\times\boldsymbol{\nu}\right] \qquad \boldsymbol{E}_{\mathrm{rad}} = c[\boldsymbol{B}_{\mathrm{rad}}(\boldsymbol{r},t)\times\boldsymbol{\nu}] \qquad \boldsymbol{\nu} = \frac{\boldsymbol{r}}{r}. \tag{9}$$

Rewriting equation (6) as

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\nabla^{n-1}\cdot\left[\dot{\mathbf{M}}^{(n)}(t_0)\times\frac{\boldsymbol{r}}{cr^2} + \mathbf{M}^{(n)}\times\frac{\boldsymbol{r}}{r^3}\right] + \frac{\mu_0}{4\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\nabla^{n-1}\cdot\left[\frac{1}{r}\dot{\mathbf{P}}^{(n)}(t_0)\right] \qquad t_0 = t - \frac{r}{c}$$

we see that the terms of the order $1/r$ are produced only by the time derivative. Finally, one obtains the following expansion for calculating the radiation [7]:

$$\boldsymbol{A}_{\mathrm{rad}}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\frac{1}{r}\sum_{n=1}^{\infty}\frac{1}{n!c^n}\boldsymbol{\nu}^{n-1}\cdot\left[\frac{\mathrm{d}^n\mathbf{M}^{(n)}(t_0)}{\mathrm{d}t^n}\times\boldsymbol{\nu}\right] + \frac{\mu_0}{4\pi}\frac{1}{cr}\sum_{n=1}^{\infty}\frac{1}{n!c^n}\boldsymbol{\nu}^{n-1}\cdot\left[\frac{\mathrm{d}^n\mathbf{P}^{(n)}(t_0)}{\mathrm{d}t^n}\right]. \tag{10}$$

This relation gives the explicit contribution of each multipole to the radiation field.

To conclude, we will note that the angular distribution of radiation resulting from (9),

$$\mathcal{I}(\boldsymbol{\nu}) = \frac{r^2}{\mu_0 c}\left(\boldsymbol{\nu}\times\frac{\partial \boldsymbol{A}_{\mathrm{rad}}}{\partial t}\right)^2$$

can be used for calculating $\mathcal{I}(\boldsymbol{\nu})$ for a point charge directly from the Liénard–Wiechert potentials [11].

3. Reduction of multipole tensors, the static case

As an exercise, which will be helpful for understanding the new features of the reduction procedure in the dynamic case, in this section we will consider the static case. We will follow the presentation given in [6].

In the static case, equations (3) and (6) become

$$\Phi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\nabla^n\cdot\left[\frac{1}{r}\mathbf{P}^{(n)}\right] = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\mathbf{P}^{(n)}\cdot\nabla^n\frac{1}{r}$$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\nabla\times\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\nabla^{n-1}\cdot\left[\frac{\mathbf{M}^{(n)}}{r}\right] = \frac{\mu_0}{4\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\nabla^{n-1}\cdot\left[\mathbf{M}^{(n)}\times\frac{\boldsymbol{r}}{r^3}\right]$$

and the electric and magnetic fields can be treated independently.

The symmetric tensor $\mathsf{P}^{(n)}$ is reduced to a symmetric traceless tensor $\boldsymbol{P}^{(n)}$ by transformations which can be written in the following form,

$$\mathsf{P}_{i_1\ldots i_n} \longrightarrow \mathsf{P}_{i_1\ldots i_n} - \lambda_{[i_1\ldots i_{n-2}}\delta_{i_{n-1}i_n]} \tag{11}$$

where $\boldsymbol{\lambda}^{(n-2)}$ is a symmetric tensor, and the index symbol $[i_1\ldots i_n]$ means the sum over all the distinct (taking into account the symmetry of $\boldsymbol{\lambda}$) permutations of indices $i_1\ldots i_n$. The components of the tensor $\boldsymbol{\lambda}$ are determined in terms of the traces of the tensor $\mathsf{P}^{(n)}$. So, for $n = 2$ we have the well-known expression

$$\lambda = \frac{1}{3}\mathsf{P}_{ii} = \frac{1}{3}\int_{\mathcal{D}} \xi^2 \rho \, \mathrm{d}^3\xi.$$

For $n = 3, 4, 5$ we have

$$\lambda_i = \frac{1}{5}\mathsf{P}_{jji} = \frac{1}{5}\int_{\mathcal{D}} \xi^2\xi_i \rho \, \mathrm{d}^3\xi \qquad \lambda_{ij} = \tfrac{1}{5}\mathsf{P}_{kkij} - \tfrac{1}{40}\mathsf{P}_{kkll}\delta_{ij}$$
$$\lambda_{ijk} = \tfrac{1}{9}\mathsf{P}_{llijk} - \tfrac{1}{126}(\mathsf{P}_{llmmi}\delta_{jk} + \mathsf{P}_{llmmj}\delta_{ki} + \mathsf{P}_{llmmk}\delta_{ij})$$

and so on. Maybe it is possible to elaborate symbolic computer programs (using, for example, REDUCE or MATHEMATICA) for generating these tensors for arbitrary orders.

Since $\Delta(1/r) = 0$, the static scalar potential Φ is invariant under the transformations (11). Indeed, by the transformation (11) we obtain

$$\Phi^{(n)}(\boldsymbol{r}) \longrightarrow \Phi^{(n)}(\boldsymbol{r}) - \frac{(-1)^n}{4\pi\varepsilon_0 n!}\partial_{i_1\ldots i_n}\left[\frac{1}{r}\lambda_{[i_1\ldots i_{n-2}}\delta_{i_{n-1}i_n]}\right]$$

where the symbol δ generates the Laplace operator Δ such that

$$\Phi^{(n)}(\boldsymbol{r}) \longrightarrow \Phi^{(n)}(\boldsymbol{r}) - \frac{(-1)^n n(n-1)}{8\pi\varepsilon_0 n!}\partial_{i_1\ldots i_{n-2}}\Delta\left[\frac{1}{r}\lambda_{i_1\ldots i_{n-2}}\right]$$
$$= \Phi^{(n)}(\boldsymbol{r}) - \frac{(-1)^n n(n-1)}{8\pi\varepsilon_0 n!}\nabla^{n-2}\cdot\left[\boldsymbol{\lambda}^{(n-2)}\Delta\frac{1}{r}\right] \qquad (r \neq 0).$$

The symmetric traceless tensor $\boldsymbol{P}^{(n)}$ can be written as [2]

$$\boldsymbol{P}^{(n)} = \frac{(-1)^n}{(2n-1)!!}\int_{\mathcal{D}} \rho(\boldsymbol{\xi})\xi^{2n+1}\nabla^n\frac{1}{\xi}\,\mathrm{d}^3\xi. \tag{12}$$

Let us now consider the nth-order term from the expansion of the vector potential

$$A_i^{(n)}(\boldsymbol{r}) = \frac{\mu_0(-1)^{n-1}}{4\pi n!}\varepsilon_{ikl}\partial_k\partial_{i_1\ldots i_{n-1}}\left(\frac{1}{r}\mathsf{M}_{i_1\ldots i_{n-1}l}\right).$$

Since the magnetic nth-order tensor $\mathsf{M}^{(n)}$ is symmetric only in the first $n-1$ indices, the total symmetrization of this tensor can be performed by the transformation

$$\mathsf{M}_{i_1\ldots i_n} \longrightarrow \mathsf{M}_{(\mathrm{sym})i_1\ldots i_n} = \mathsf{M}_{i_1\ldots i_n} - \frac{1}{n}\sum_{\lambda=1}^{n-1}\left[\mathsf{M}_{i_1\ldots i_n} - \mathsf{M}_{i_1\ldots i_{\lambda-1}i_{\lambda+1}\ldots i_{n-1}i_n i_\lambda}\right]$$
$$= \mathsf{M}_{i_1\ldots i_n} - \frac{1}{n}\sum_{\lambda=1}^{n-1}\left[\mathsf{M}_{i_1\ldots i_{\lambda-1}i_{\lambda+1}\ldots i_{n-1}i_\lambda i_n} - \mathsf{M}_{i_1\ldots i_{\lambda-1}i_{\lambda+1}\ldots i_{n-1}i_n i_\lambda}\right]$$

or

$$\mathsf{M}_{i_1\ldots i_n} \longrightarrow \mathsf{M}_{(\mathrm{sym})i_1\ldots i_n} = \mathsf{M}_{i_1\ldots i_n} - \frac{1}{n}\sum_{\lambda=1}^{n-1}\left[\mathsf{M}^{(\lambda)}_{i_1\ldots i_{n-1}i_\lambda i_n} - \mathsf{M}^{(\lambda)}_{i_1\ldots i_{n-1}i_n i_\lambda}\right] \tag{13}$$

where we use the notation

$$f^{(\lambda)}_{i_1\ldots i_k} = f_{i_1\ldots i_{\lambda-1} i_{\lambda+1}\ldots i_k}. \tag{14}$$

Defining the $(n-1)$th-order tensor $\mathsf{N}^{(n-1)}$ by its Cartesian components

$$\mathsf{N}_{i_1\ldots i_{n-1}} = \varepsilon_{i_{n-1}ps}\mathsf{M}_{i_1\ldots i_{n-2}ps} = \frac{n}{n+1}\varepsilon_{i_{n-1}ps}\int_{\mathcal{D}} \xi_{i_1}\cdots\xi_{i_{n-2}}\xi_p(\boldsymbol{\xi}\times\boldsymbol{j})_s\,\mathrm{d}^3\xi$$
$$= \frac{n}{n+1}\int_{\mathcal{D}} \xi_{i_1}\cdots\xi_{i_{n-2}}[\boldsymbol{\xi}\times(\boldsymbol{\xi}\times\boldsymbol{j})]_{i_{n-1}}\,\mathrm{d}^3\xi$$

the transformation (13) can be written as

$$\mathsf{M}_{i_1\ldots i_n} \longrightarrow \mathsf{M}_{(\mathrm{sym})i_1\ldots i_n} = \mathsf{M}_{i_1\ldots i_n} - \frac{1}{n}\sum_{\lambda=1}^{n-1}\varepsilon_{i_\lambda i_n q}\mathsf{N}^{(\lambda)}_{i_1\ldots i_{n-1}q}.$$

Here we again use the notation (14). We write explicitly the modification of the term $A_i^{(n)}(\boldsymbol{r})$ induced by the substitution (13):

$$A_i^{(n)}(\boldsymbol{r}) \longrightarrow A_i^{(n)}(\boldsymbol{r}) - \frac{\mu_0(-1)^{n-1}}{4\pi n!n}\varepsilon_{ikl}\partial_k\partial_{i_1\ldots i_{n-1}}\left[\frac{1}{r}\sum_{\lambda=1}^{n-1}\varepsilon_{i_\lambda lq}\mathsf{N}^{(\lambda)}_{i_1\ldots i_{n-1}q}\right]$$
$$= A_i^{(n)}(\boldsymbol{r}) + \frac{\mu_0(-1)^{n-1}}{4\pi n!n}\sum_{\lambda=1}^{n-1}\varepsilon_{ikl}\varepsilon_{i_\lambda ql}\partial_k\partial_{i_\lambda}\partial^{(\lambda)}_{i_1\ldots i_{n-1}}\left[\frac{1}{r}\mathsf{N}^{(\lambda)}_{i_1\ldots i_{n-1}q}\right]$$

and, because $\varepsilon_{ikl}\varepsilon_{i_\lambda ql} = \delta_{ii_\lambda}\delta_{kq} - \delta_{iq}\delta_{ki_\lambda}$,

$$A_i^{(n)}(\boldsymbol{r}) \longrightarrow A_i^{(n)}(\boldsymbol{r}) + \frac{\mu_0(-1)^{n-1}}{4\pi n!n}\sum_{\lambda=1}^{n-1}\left\{\partial_k\partial_i\left[\boldsymbol{\nabla}^{n-1}\cdot\frac{1}{r}\mathbf{N}^{(n-1)}\right]_k - \partial_k\partial_k\left[\boldsymbol{\nabla}^{n-1}\cdot\frac{1}{r}\mathbf{N}^{(n-1)}\right]_i\right\}$$

such that

$$\boldsymbol{A}(\boldsymbol{r}) \longrightarrow \boldsymbol{A}(\boldsymbol{r}) + \frac{\mu_0(-1)^n(n-1)}{4\pi n!n}\boldsymbol{\nabla}^{n-2}\cdot\Delta\left[\frac{1}{r}\mathbf{N}^{(n-1)}\right] + \boldsymbol{\nabla}\psi^{(n)}(\boldsymbol{r}) \tag{16}$$

where

$$\psi^{(n)}(\boldsymbol{r}) = \frac{\mu_0(-1)^{n-1}(n-1)}{4\pi n!n}\boldsymbol{\nabla}^{n-1}\cdot\left[\frac{1}{r}\mathbf{N}^{(n-1)}\right] \tag{17}$$

and, using $\Delta(1/r) = 0$, we see that the transformation (16) is a gauge transformation.

The reduction of $\mathbf{M}^{(n)}_{(\mathrm{sym})}$ to a symmetric traceless tensor $\boldsymbol{M}^{(n)}$ is achieved by the transformation

$$\mathsf{M}_{(\mathrm{sym})i_1\ldots i_n} \longrightarrow \mathsf{M}_{(\mathrm{sym})i_1\ldots i_n} - \chi_{[i_1\ldots i_{n-2}}\delta_{i_{n-1}i_n]} \tag{18}$$

where the tensors $\boldsymbol{\xi}^{(n-2)}$ are expressed by the traces of the tensors $\mathbf{M}^{(n)}_{(\mathrm{sym})}$ by the same procedure as in the case of tensors $\boldsymbol{\lambda}^{(n-2)}$ from equation (11). Then

$$A_i + \partial_i\psi \longrightarrow A_i + \partial_i\psi - \frac{\mu_0(-1)^{n-1}}{4\pi n!}\varepsilon_{ikl}\partial_k\partial_{i_1\ldots i_{n-1}}\left[\frac{1}{r}\chi_{[i_1\ldots i_{n-2}}\delta_{i_{n-1}l]}\right]. \tag{19}$$

In the last sum above, we have $n-1$ terms in which l is an index of δ, leading to the null operator $\varepsilon_{ikl}\partial_k\partial_l$. For the remaining $(n-1)(n-2)/2$ terms, in which l is an index of the χ-components (in an arbitrary position, due to the symmetry of χ), the symbol δ generates the Laplace operator Δ, which gives again zero in the static case. Thus, the static vector potential is invariant under the transformation (18). Such a reduction of the magnetic multipole tensors was given in [6] and was also given by Gonzales *et al* [8], who used a different procedure.

We note that the symmetric traceless tensor $\boldsymbol{M}^{(n)}$ can be expressed as [6]

$$M_{i_1\ldots i_n} = \frac{(-1)^{n+1}}{(n+1)(2n-1)!!}\sum_{\lambda=1}^{n}\int_{\mathcal{D}}\xi^{2n+1}(\boldsymbol{j}\times\nabla)_{i_\lambda}\partial^{(\lambda)}_{i_1\ldots i_n}\frac{1}{r}. \tag{20}$$

Using the symmetric traceless tensor $\boldsymbol{M}^{(n)}$ in the expansion of $\boldsymbol{A}$ it is readily seen that, outside the domain $\mathcal{D}$, the magnetic field $\boldsymbol{B} = -\nabla\Psi$, where

$$\Psi(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\boldsymbol{M}^{(n)}\cdot\nabla^n\frac{1}{r} \tag{21}$$

is the magnetic scalar potential. The multipole expansion of the magnetostatic field using the magnetic scalar potential was given by Gray [12].

Using expansion (21) for the scalar magnetic potential Ψ, similar formulae for the electrostatic and magnetostatic interactions can be written. For example, using the symmetric traceless electric multipole, tensor $\boldsymbol{P}^{(n)}$, the expansion of the electric field is given by [6–8]

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{(2n-1)!!}{n!r^{n+2}}[(2n+1)(\boldsymbol{P}^{(n)}\cdot\boldsymbol{\nu}^n)\boldsymbol{\nu} - n(\boldsymbol{P}^{(n)}\cdot\boldsymbol{\nu}^{n-1})]$$

and a similar formula can be written for the magnetic field.

As a second example, let us consider the interaction energy between two static electric charge distributions ρ and ρ' with disjoint supports $\operatorname{supp}\rho\subset\mathcal{D}$, $\operatorname{supp}\rho'\subset\mathcal{D}'$ and $\mathcal{D}\cap\mathcal{D}'=\emptyset$. The electrostatic interaction energy can be written as

$$W^{(e)}_{\mathrm{int}} = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n}{m!n!}(\boldsymbol{P}^{(n)}\cdot\nabla^n)(\boldsymbol{P}'^{(m)}\cdot\nabla^m)\frac{1}{r}.$$

Here the nm term represents the interaction energy $W^{(n,m)}_{\mathrm{int}}$ between a point-like nth-order electric multipole, placed in the origin point $O\in\mathcal{D}$, and a point-like mth-order electric multipole, placed in $O'\in\mathcal{D}'$. In these formulae, $\boldsymbol{r}$ is the position vector of O'. By recursive induction one may show that [6, 7]

$$W^{(n,m)}_{\mathrm{int}} = \frac{(-1)^m}{4\pi\varepsilon_0 m!n!r^{n+m+1}} \times\sum_{k=0}^{\min(n,m)}(-1)^k k![2(m+n-k)-1]!!C^k_n C^k_m(\boldsymbol{\nu}^{n-k}\cdot\boldsymbol{P}^{(n)})\cdot(\boldsymbol{P}'^{(m)}\cdot\boldsymbol{\nu}^{m-k})$$

and, obviously, similar expressions are obtained in the magnetostatic case.

4. Reduction of multipole tensors, the dynamic case

We look for transformations of the first N electric and magnetic multipole tensors which satisfy the condition that the corresponding sums for $\boldsymbol{E}$ and $\boldsymbol{B}$, i.e. $\sum_{n\leqslant N}\boldsymbol{E}^{(n)}$, $\sum_{n\leqslant N}\boldsymbol{B}^{(n)}$, remain unchanged. This condition will be fulfilled if the corresponding sums for the potentials Φ and $\boldsymbol{A}$ are modified only by a gauge transformation.

Since, as we will see below, the reduction of the multipole tensors of a given order n generates modifications of the multipole tensors of the lower orders, we suppose only that $\mathsf{P}^{(n)}$ is fully symmetric and $\mathsf{M}^{(n)}$ is symmetric in the first $n-1$ indices.

Let us first consider the modification of the vector potential $\boldsymbol{A}$ by the transformations (13) which lead to a fully symmetric magnetic multipole tensor $\mathsf{M}^{(n)}_{(\mathrm{sym})}$. Due to the symmetry in the first $n-2$ indices of the tensor (15) and

$$\Box\frac{f(t-r/c)}{r} = 0 \qquad r\neq 0$$

the new vector potential is given by an expression which is a generalization of equation (16),

$$\boldsymbol{A} \longrightarrow \boldsymbol{A}' = \boldsymbol{A} + \nabla\psi^{(n)} + \frac{\mu_0(-1)^n(n-1)}{4\pi c^2 n!n}\nabla^{n-2}\cdot\left[\frac{1}{r}\ddot{\mathsf{N}}^{(n-1)}(t_0)\right]$$

where $\psi^{(n)}$ is given by equation (17). Here

$$\mathsf{N}_{i_1\ldots i_{n-1}} = \varepsilon_{i_{n-1}ps}\mathsf{M}_{i_1\ldots i_{n-2}ps} \tag{22}$$

and the representation by the integral from (15) is verified only if $\mathsf{M}^{(n)}$ is given by relation (5).

We may obtain a gauge transformation of the scalar potential represented by $\Phi \longrightarrow \Phi - \partial\psi^{(n)}/\partial t$ changing the $(n-1)$th-order electric multipole tensor:

$$\mathsf{P}^{(n-1)} \longrightarrow \mathsf{P}^{(n-1)} + \delta\mathsf{P}^{(n-1)} : \delta\mathsf{P}^{(n-1)} = -\frac{n-1}{c^2n^2}\dot{\mathsf{N}}^{(n-1)}. \tag{23}$$

On the other hand, the transformation (23) produces an alteration of the vector potential $\boldsymbol{A}'$:

$$\boldsymbol{A}' \longrightarrow \boldsymbol{A}' - \frac{\mu_0(-1)^n(n-1)}{4\pi c^2 n!n}\nabla^{n-2}\cdot\left[\frac{1}{r}\ddot{\mathsf{N}}^{(n-1)}(t_0)\right]$$

the resulting transformation of the potentials being $\boldsymbol{A} \longrightarrow \boldsymbol{A} + \nabla\psi^{(n)}$, $\Phi \longrightarrow \Phi - \partial\psi^{(n)}/\partial t$ such that to the transformations (13) and (23) corresponds a gauge transformation of the potentials Φ and $\boldsymbol{A}$ with $\psi^{(n)}$ given by equation (17).

For the sake of simplicity, in the following, we will use the same notations, Φ and $\boldsymbol{A}$, for the new potentials.

After the reduction of the magnetic tensor $\mathsf{M}^{(n)}$ to a symmetric one, we have to perform the reduction to a traceless tensor $\tilde{\mathsf{M}}^{(n)}$ by transformations of the type (18). The alteration of the vector potential is represented by equations (19) with $\chi(t - r/c)$ and repeating the considerations from the static case (this time $\Delta[\chi(t_0)/r] = \ddot{\chi}/c^2$) one obtains

$$\begin{aligned}
A_i^{(n)}(\boldsymbol{r},t) \longrightarrow A_i^{(n)}(\boldsymbol{r},t) &- \frac{\mu_0(-1)^{n-1}}{4\pi n!}\frac{(n-1)(n-2)}{2}\varepsilon_{ikl}\partial_k\partial_{i_1\ldots i_{n-3}}\Delta\left[\frac{1}{r}\chi_{i_1\ldots i_{n-3}l}(t_0)\right] \\
&= A_i^{(n)}(\boldsymbol{r},t) - \frac{\mu_0(-1)^{n-1}(n-1)(n-2)}{8\pi n!c^2}\varepsilon_{ikl}\partial_k\left\{\nabla^{n-3}\cdot\left[\frac{1}{r}\ddot{\chi}^{(n-2)}(t_0)\right]\right\}_l \\
&= A_i^{(n)}(\boldsymbol{r},t) - \frac{\mu_0(-1)^{n-1}(n-1)(n-2)}{8\pi n!c^2}\left\{\nabla\times\left[\nabla^{n-3}\cdot\left(\frac{1}{r}\ddot{\chi}^{(n-2)}(t_0)\right)\right]\right\}_i
\end{aligned}$$

and

$$\boldsymbol{A} \longrightarrow \boldsymbol{A} - \frac{\mu_0(-1)^{n-1}(n-1)(n-2)}{8\pi c^2 n!}\nabla\times\left\{\nabla^{n-3}\cdot\left[\frac{1}{r}\ddot{\chi}^{(n-2)}\right]\right\}. \tag{24}$$

The supplementary term from equation (24) can be set off by the following transformation of $\mathsf{M}^{(n-2)}$:

$$\mathsf{M}^{(n-2)} \longrightarrow \mathsf{M}'^{(n-2)} = \mathsf{M}^{(n-2)} + \frac{(n-2)}{2c^2n}\ddot{\chi}^{(n-2)}.$$

The last step consists in the reduction of the symmetric nth-order electric multipole tensor $\mathsf{P}^{(n)}$ to a traceless one by transformations of the type (11). The alterations of the potentials are given by

$$\Phi(\boldsymbol{r},t) \longrightarrow \Phi(\boldsymbol{r},t) - \frac{(-1)^n}{4\pi\varepsilon_0 n!}\partial_{i_1\ldots i_n}\left[\frac{1}{r}\lambda_{[i_1\ldots i_{n-2}}\delta_{i_{n-1}i_n]}(t_0)\right]$$

$$= \Phi(\boldsymbol{r},t) - \frac{(-1)^n n(n-1)}{8\pi\varepsilon_0 n!}\boldsymbol{\nabla}^{n-2}\cdot\left\{\Delta\left[\frac{1}{r}\boldsymbol{\lambda}^{(n-2)}(t_0)\right]\right\}$$

$$= \Phi(\boldsymbol{r},t) - \frac{(-1)^n n(n-1)}{8\pi\varepsilon_0 c^2 n!}\boldsymbol{\nabla}^{n-2}\cdot\left[\frac{1}{r}\ddot{\boldsymbol{\lambda}}^{(n-2)}(t_0)\right]$$

$$\boldsymbol{A}(\boldsymbol{r},t) \longrightarrow \boldsymbol{A}(\boldsymbol{r},t) - \frac{\mu_0(-1)^{n-1}}{4\pi n!}\partial_{i_1\ldots i_{n-1}}\left[\frac{1}{r}\dot{\lambda}_{[i_1\ldots i_{n-2}}(t_0)\delta_{i_{n-1}i]}\right]\boldsymbol{e}_i$$

$$= \boldsymbol{A}(\boldsymbol{r},t) - \frac{\mu_0(-1)^{n-1}(n-1)(n-2)}{8\pi n!}\partial_{i_1\ldots i_{n-3}}\left\{\Delta\left[\frac{1}{r}\dot{\lambda}^{(n-2)}_{i_1\ldots i_{n-3}i}(t_0)\right]\right\}\boldsymbol{e}_i$$

$$- \frac{\mu_0(-1)^{n-1}(n-1)}{4\pi n!}\partial_i\partial_{i_1\ldots i_{n-2}}\left[\frac{1}{r}\dot{\lambda}^{(n-2)}_{i_1\ldots i_{n-2}}(t_0)\right]\boldsymbol{e}_i$$

$$= \boldsymbol{A}(\boldsymbol{r},t) - \frac{\mu_0(-1)^{n-1}(n-1)(n-2)}{8\pi c^2 n!}\boldsymbol{\nabla}^{n-3}\cdot\left[\frac{1}{r}\dddot{\boldsymbol{\lambda}}^{(n-2)}(t_0)\right]$$

$$- \frac{\mu_0(-1)^{n-1}(n-1)}{4\pi n!}\boldsymbol{\nabla}\cdot\left\{\boldsymbol{\nabla}^{n-2}\cdot\left[\frac{1}{r}\dot{\boldsymbol{\lambda}}^{(n-2)}(t_0)\right]\right\}.$$

We perform now the transformation

$$\mathsf{P}^{(n-2)} \longrightarrow \mathsf{P}^{(n-2)} + \frac{a}{2c^2}\ddot{\boldsymbol{\lambda}}^{(n-2)} \tag{25}$$

with a parameter a which will be chosen adequately. The results of the transformations (11) and (25) are

$$\Phi \longrightarrow \Phi + \frac{(-1)^n(a-1)}{8\pi\varepsilon_0 c^2(n-2)!}\boldsymbol{\nabla}^{n-2}\cdot\left[\frac{1}{r}\ddot{\boldsymbol{\lambda}}^{(n-2)}(t_0)\right]$$

$$\boldsymbol{A} \longrightarrow \boldsymbol{A} - \frac{\mu_0(-1)^{n-1}(n-1)}{4\pi n!}\boldsymbol{\nabla}\cdot\left\{\boldsymbol{\nabla}^{n-2}\cdot\left[\frac{1}{r}\dot{\boldsymbol{\lambda}}^{(n-2)}(t_0)\right]\right\}$$

$$+ \left(a - \frac{n-2}{n}\right)\frac{\mu_0(-1)^{n-1}}{8\pi c^2(n-2)!}\boldsymbol{\nabla}^{n-3}\cdot\left[\frac{1}{r}\dddot{\boldsymbol{\lambda}}^{(n-2)}(t_0)\right].$$

Choosing

$$a = \frac{n-2}{n}$$

one obtains the following gauge transformations

$$\boldsymbol{A} \longrightarrow \boldsymbol{A} + \boldsymbol{\nabla}\psi'^{(n)} \qquad \Phi \longrightarrow \Phi - \frac{\partial}{\partial t}\psi'^{(n)}$$

with

$$\psi'^{(n)} = \frac{\mu_0(-1)^n(n-1)}{4\pi n!}\boldsymbol{\nabla}^{n-2}\cdot\left[\frac{1}{r}\dot{\boldsymbol{\lambda}}^{(n-2)}(t_0)\right].$$

The transformation (23) alters the symmetry properties of the $(n-1)$th electric multipole tensor because of $\delta\mathsf{P}^{(n-1)}$ which is symmetric only in the first $n-2$ indices. To restore the full symmetry of the $(n-1)$th electric moment, we perform the transformation of $\mathsf{P}^{(n-1)} + \delta\mathsf{P}^{(n-1)}$ by

$$\delta\mathsf{P}_{i_1\ldots i_{n-1}} \longrightarrow \delta\mathsf{P}_{i_1\ldots i_{n-1}} + \frac{1}{c^2n^2}\sum_{\lambda=1}^{n-2}\left[\dot{N}_{i_1\ldots i_n-1} - \dot{N}^{(\lambda)}_{i_1\ldots i_{n-1}i_\lambda}\right].$$

By introducing the tensor $\boldsymbol{\mathcal{N}}^{(n-2)}$ with components

$$\mathcal{N}_{i_1\ldots i_{n-2}} = \varepsilon_{i_{n-2}ps}\mathsf{N}_{i_1\ldots i_{n-3}ps}$$

the transformation of $\delta\mathsf{P}^{(n-1)}$ can be written as

$$\delta\mathsf{P}_{i_1\ldots i_{n-1}} \longrightarrow \delta\mathsf{P}_{i_1\ldots i_{n-1}} + \frac{1}{c^2n^2}\sum_{\lambda=1}^{n-2}\varepsilon_{i_\lambda i_{n-1}q}\dot{\mathcal{N}}^{(\lambda)}_{i_1\ldots i_{n-2}q}. \tag{26}$$

If $\mathsf{M}^{(n)}$ is given by equation (5), we can write

$$\mathcal{N}_{i_1\ldots i_{n-3}i_{n-2}} = -\frac{n}{n+1}\int_{\mathcal{D}}\xi^2\xi_{i_1}\ldots\xi_{i_{n-3}}(\boldsymbol{\xi}\times\boldsymbol{j})_{i_{n-2}}\mathrm{d}^3\xi.$$

It is a simple matter to see that the transformations (26) do not alter the scalar potential Φ, but the vector potential is transformed by

$$\boldsymbol{A} \longrightarrow \boldsymbol{A} - \frac{\mu_0(-1)^n(n-2)}{4\pi c^2 n!n}\nabla\times\left\{\nabla^{n-3}\cdot\left[\frac{1}{r}\ddot{\boldsymbol{\mathcal{N}}}^{(n-2)}(t_0)\right]\right\}.$$

This alteration of the vector potential may be set off by the transformation

$$\mathsf{M}'^{(n-2)} \longrightarrow \mathsf{M}'^{(n-2)} - \frac{(n-2)}{c^2n^2(n-1)}\ddot{\boldsymbol{\mathcal{N}}}^{(n-2)} \tag{27}$$

which preserves the symmetry properties of $\mathsf{M}^{(n-2)}$.

The transformations (26) and (27) must be considered for applying the procedure of reduction in the $(n-1)$th and $(n-2)$th orders, respectively.

5. Conclusions

We summarize the results of the last section by the following statements.

(1) The reduction of the magnetic nth-order multipole tensor to a symmetric traceless one by the transformation

$$\mathsf{M}_{i_1\ldots i_n} \longrightarrow \mathsf{M}_{i_1\ldots i_n} - \frac{1}{n}\sum_{\lambda=1}^{n-1}\varepsilon_{i_\lambda i_n q}\mathsf{N}^{(\lambda)}_{i_1\ldots i_{n-1}q} - \chi_{[i_1\ldots i_{n-2}}\delta_{i_{n-1}i_n]}$$

together the modifications of the electric $(n-1)$th order and of the magnetic $(n-2)$th-order multipole tensors

$$\mathsf{P}_{i_1\ldots i_{n-1}} \longrightarrow \mathsf{P}_{i_1\ldots i_{n-1}} - \frac{n-1}{c^2n^2}\dot{N}_{i_1\ldots i_{n-1}} \tag{28}$$

$$\mathsf{M}_{i_1\ldots i_{n-2}} \longrightarrow \mathsf{M}_{i_1\ldots i_{n-2}} + \frac{(n-2)}{2c^2n}\ddot{\chi}_{i_1\ldots i_{n-2}} \tag{29}$$

leads to a gauge transformation of the potentials Φ and $\boldsymbol{A}$.

(2) The reduction of the electric nth-order multipole tensor to a symmetric traceless one

$$\mathsf{P}_{i_1\ldots i_n} \longrightarrow \mathsf{P}_{i_1\ldots i_n} - \lambda_{[i_1\ldots i_{n-2}}\delta_{i_{n-1}i_n]}$$

together the transformation of the electric $(n-2)$th-order multipole tensor

$$\mathsf{P}_{i_1\ldots i_{n-2}} \longrightarrow \mathsf{P}_{i_1\ldots i_{n-2}} + \frac{n-2}{2nc^2}\ddot{\lambda}_{i_1\ldots i_{n-2}} \tag{30}$$

leads also to a gauge transformation of the potentials.

The tensor $\mathsf{N}^{(n-1)}$ is given by equation (22), $\boldsymbol{\chi}^{(n-2)}$ is expressed in terms of the traces of the tensor $\mathsf{M}^{(n)}_{(\mathrm{sym})}$ and $\boldsymbol{\lambda}^{(n-2)}$ is expressed in terms of the traces of $\mathsf{P}^{(n)}$.

The reduction procedure described above do not alter the sums

$$\sum_{k\leqslant n}\boldsymbol{E}^{(k)} \qquad \sum_{k\leqslant n}\boldsymbol{B}^{(k)}$$

but some terms from these sums are altered, this fact being physically irrelevant. Concerning the potentials, the corresponding sums of multipole terms are modified only by gauge transformations with ψ-functions which are solutions of the homogeneous wave equation such that the Lorenz condition is preserved[1].

The procedure applied for the order n can be applied in all the lower orders. We point out that the transformations (28), (29) and (30) together with the transformations (26) and (27) do not change the symmetry properties used in the nth-order; hence, the procedure may be applied step by step for all the lower orders.

If we begin the reduction from a given order N, then the results of the reductions of $\mathsf{P}^{(N)}$ and $\mathsf{M}^{(N)}$ are the tensors $\boldsymbol{P}^{(N)}$ and $\boldsymbol{M}^{(N)}$ given by equations (12) and (20). For $n < N$, the nth-order reduced multipole tensors may differ from $\boldsymbol{P}^{(n)}$ and $\boldsymbol{M}^{(n)}$ by terms induced by the reductions from the previous steps. These last terms give contributions to the potentials and fields expressed by toroidal moments and mean radii of various orders.

We give a simple example of such a reduction and we will see how the toroidal moments appear as a result of such an approach.

By reducing the magnetic moments beginning from the third order and the electric ones from the fourth order, and considering the final results for the electric multipole tensors, we obtain

$$\mathsf{P}_{ijkl} \rightarrow P_{ijkl} \qquad \mathsf{P}_{ijk} \rightarrow P_{ijk}$$

but, concerning the electric dipolar and quadrupolar moments, the following reduced tensors are obtained:

$$\begin{aligned}\tilde{\mathsf{P}}_i &= P_i + \frac{1}{6c^2}\ddot{\lambda}_i - \frac{1}{4c^2}\dot{\mathsf{N}}_i\\ \tilde{\mathsf{P}}_{ij} &= P_{ij} - \frac{1}{9c^2}(\dot{\mathsf{N}}_{ij} + \dot{\mathsf{N}}_{ji}) + \frac{1}{4c^2}\left(\ddot{\lambda}_{ij} - \frac{1}{3}\ddot{\lambda}_{kk}\delta_{ij}\right).\end{aligned} \tag{31}$$

Here

$$\begin{aligned}&\mathsf{N}_{ik} = \frac{3}{4}\int_{\mathcal{D}}\xi_i[\boldsymbol{\xi}\times(\boldsymbol{\xi}\times\boldsymbol{j})]_k\,\mathrm{d}^3\xi \qquad \mathsf{N}_i = \frac{2}{3}\int_{\mathcal{D}}[\boldsymbol{\xi}\times(\boldsymbol{\xi}\times\boldsymbol{j})]_i\,\mathrm{d}^3\xi\\ &\lambda_{ij} = \frac{1}{7}\int_{\mathcal{D}}\xi^2\xi_i\xi_j\rho\,\mathrm{d}^3\xi - \frac{1}{70}\int_{\mathcal{D}}\xi^4\rho\,\mathrm{d}^3\xi\,\delta_{ij} \qquad \lambda_i = \frac{1}{5}\int_{\mathcal{D}}\xi^2\xi_i\rho\,\mathrm{d}^3\xi\\ &P_i = \int_{\mathcal{D}}\xi_i\rho\,\mathrm{d}^3\xi \qquad P_{ij} = \int_{\mathcal{D}}\left(\xi_i\xi_j - \frac{1}{3}\xi^2\delta_{ij}\right)\rho\,\mathrm{d}^3\xi.\end{aligned} \tag{32}$$

[1] The condition $\nabla\,.\,\boldsymbol{A} + \varepsilon_0\mu_0\partial\Phi/\partial t = 0$ was given in 1867 by the Danish scientist Ludwig Lorenz, not by the Dutch scientist H A Lorentz, but this constraint was generally misattributed to H A Lorentz (see [15]). I am very grateful to an anonymous referee for bringing this to my attention.

Inserting (32) in (31) and taking into account the continuity equation verified by ρ and $\boldsymbol{j}$, we obtain

$$\tilde{\mathsf{P}}_i = P_i - \frac{1}{c^2}\dot{T}_i \qquad \tilde{\mathsf{P}}_{ij} = P_{ij} - \frac{1}{c^2}\dot{T}_{ij} \tag{33}$$

where

$$\mathsf{T}_i = \frac{1}{10}\int_{\mathcal{D}} [\xi_i(\boldsymbol{\xi}\cdot\boldsymbol{j}) - 2\xi^2 j_i]\,\mathrm{d}^3\xi \tag{34}$$

$$\mathsf{T}_{ik} = \frac{1}{42}\int_{\mathcal{D}} [4\xi_i\xi_k(\boldsymbol{\xi}\cdot\boldsymbol{j}) - 5\xi^2(\xi_i j_k + \xi_k j_i) + 2\xi^2(\boldsymbol{\xi}\cdot\boldsymbol{j})\delta_{ik}]\,\mathrm{d}^3\xi \tag{35}$$

are the toroid dipole and quadrupole tensors, respectively [13, 14].

Using (33) in equation (10), we can calculate the toroidal multipole contributions to the radiation field.

Applying the reduction for $n \to \infty$, it is clear that one obtains infinite sums representing the multipole tensors and, maybe, it is a formidable task to give a general rule in Cartesian coordinates. As is known, this task is accomplished in spherical coordinates obtaining the multipole moments represented by integrals of the spherical Bessel functions.

Acknowledgments

This article is dedicated to the memory of the late Professor Gh A Mezincescu. The author is grateful for his valuable criticisms and unforgettable comments regarding the present paper and other works also. I thank Dr Cr Stoica for reading the manuscript and for many helpful comments.

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